

# BI-LEGENDRIAN STRUCTURES AND PARACONTACT GEOMETRY

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**ABSTRACT.** We study the interplays between paracontact geometry and the theory of bi-Legendrian manifolds. We interpret the bi-Legendrian connection of a bi-Legendrian manifold  $M$  as the paracontact connection of a canonical paracontact structure induced on  $M$  and then we discuss many consequences of that result both for bi-Legendrian and for paracontact manifolds, as a classification of paracontact metric structures. Finally new classes of examples of paracontact manifolds are presented.

## 1. INTRODUCTION

A bi-Legendrian manifold is by definition a contact manifold  $(M, \eta)$  foliated by two transversal Legendrian foliations  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . More generally, if  $M$  is endowed with a pair of transversal, not necessarily integrable Legendrian distributions, we speak of an almost bi-Legendrian manifold. The study of such structures is rather recent in literature, being started in the 90's by the works of M. Y. Pang, P. Libermann, N. Jayne et al. on Legendrian foliations ([17], [19], [20]).

In this note we study some properties of bi-Legendrian manifolds. In particular we recognize that the theory of bi-Legendrian manifolds is closely linked to paracontact geometry. We recall that paracontact manifolds are semi-Riemannian manifolds which can be viewed as the odd dimensional counterpart of paracomplex manifolds (see § 2.2 for a precise definition). These manifolds were introduced by S. Kaneyuki in [18] and then studied by other authors. More recently, there seems to be an increasing interest in paracontact geometry, and in particular in para-Sasakian geometry, due to its links to the more consolidated theory of para-Kähler manifolds and to their role in geometry and mathematical physics (cf. e.g. [1], [11], [12], [13], [14]). Many progresses in that subject have been reached by some very recent papers of D. V. Alekseevski, S. Ivanov, S. Zamkovoy and their collaborators ([2], [16], [22]). In particular in [22] a complete arrangement of all the theory is obtained and it is introduced a canonical connection, called paracontact connection, which reveals to be very useful in the study of paracontact manifolds.

The main result of this paper is that given an almost bi-Legendrian structure on a contact manifold  $(M, \eta)$ , it is induced on  $M$  a canonical paracontact metric structure, and conversely if we start from a paracontact metric manifold  $(M, \psi, \xi, \eta, g)$ , one can construct on  $M$  a canonical almost bi-Legendrian structure. This result has many consequences, arising from the interplays between these two geometric structures. In particular we are able to find new properties and new examples

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of both bi-Legendrian and paracontact manifolds. More precisely we provide a classification of paracontact metric structures based on the Pang's classification of Legendrian foliations [20] and prove that, under some natural assumptions of integrability, the canonical paracontact connection of a paracontact metric manifold coincides with the bi-Legendrian connection of the induced almost bi-Legendrian structure, introduced in [7]. This last result yields a vanishing phenomenon of certain characteristic classes in a para-Sasakian manifold, providing an obstruction to the existence of this structure. Moreover, we find some results on the curvature of the canonical paracontact connection of a para-Sasakian manifold and characterize flat para-Sasakian manifolds.

## 2. PRELIMINARIES

**2.1. Legendrian foliations.** A *contact manifold* is a  $(2n+1)$ -dimensional smooth manifold  $M$  which admits a 1-form  $\eta$  satisfying  $\eta \wedge (d\eta)^n \neq 0$  everywhere on  $M$ . It is well known that given  $\eta$  there exists a unique vector field  $\xi$ , called *Reeb vector field*, such that  $i_\xi \eta = 1$  and  $i_\xi d\eta = 0$ . We denote by  $\mathcal{D}$  the  $2n$ -dimensional distribution defined by  $\ker(\eta)$ , called the *contact distribution*. It is easy to see that the Reeb vector field is an infinitesimal automorphism with respect to the contact distribution and the tangent bundle of  $M$  splits as the direct sum  $TM = \mathcal{D} \oplus \mathbb{R}\xi$ . A Riemannian metric  $G$  on  $M$  is an *associated metric* for a contact form  $\eta$  if  $G(X, \xi) = \eta(X)$  for all  $X \in \Gamma(TM)$  and there exists a tensor field  $\phi$  of type  $(1, 1)$  on  $M$  such that  $\phi^2 = -I + \eta \otimes \xi$  and  $d\eta(X, Y) = G(X, \phi Y)$  for all  $X, Y \in \Gamma(TM)$ . From the previous conditions it easily follows that  $\phi\xi = 0$ ,  $\eta \circ \phi = 0$  and  $\phi|_{\mathcal{D}}$  is an isomorphism. We refer to  $(\phi, \xi, \eta, G)$  as a *contact metric structure* and to  $M$  endowed with such a structure as a *contact metric manifold*. A contact metric structure such that the Levi-Civita connection satisfies

$$(2.1) \quad (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$$

is said to be a *Sasakian manifold*. For more details we refer the reader to [5].

Note that the condition  $\eta \wedge (d\eta)^n \neq 0$  implies that the contact distribution is never integrable. One can prove that the maximal dimension of an integrable subbundle of  $\mathcal{D}$  is  $n$ . This motivates the following definition.

**Definition 2.1.** A Legendrian distribution on a contact manifold  $(M, \eta)$  is an  $n$ -dimensional subbundle  $L$  of the contact distribution such that  $d\eta(X, X') = 0$  for all  $X, X' \in \Gamma(L)$ . When  $L$  is integrable, it defines a Legendrian foliation of  $(M, \eta)$ . Equivalently, a Legendrian foliation of  $(M, \eta)$  is a foliation of  $M$  whose leaves are  $n$ -dimensional  $C$ -totally real submanifolds of  $(M, \eta)$ .

*Remark 2.2.* Note that when  $L$  is involutive the condition that  $d\eta(X, X') = 0$  for all  $X, X' \in \Gamma(L)$  is unnecessary. Indeed by the integrability of  $L$  one has  $2d\eta(X, X') = X(\eta(X')) - X'(\eta(X)) - \eta([X, X']) = 0$ .

Legendrian foliations have been extensively investigated in recent years from various points of views. In particular M. Y. Pang provided a classification of Legendrian foliations by means of a bilinear symmetric form  $\Pi_{\mathcal{F}}$  on the tangent bundle of the foliation, defined by  $\Pi_{\mathcal{F}}(X, X') = -(\mathcal{L}_X \mathcal{L}_{X'} \eta)(\xi)$ . He called a Legendrian foliation  $\mathcal{F}$  *non-degenerate*, *degenerate* or *flat* according to the circumstance that the bilinear form  $\Pi_{\mathcal{F}}$  is non-degenerate, degenerate or vanishes identically, respectively. A geometrical interpretation of this classification is given in the following lemma.

**Lemma 2.3** ([17]). *Let  $(M, \phi, \xi, \eta, g)$  be a contact metric manifold foliated by a Legendrian foliation  $\mathcal{F}$ . Then*

- (i)  $\mathcal{F}$  is flat if and only if  $[\xi, X] \in \Gamma(T\mathcal{F})$  for all  $X \in \Gamma(T\mathcal{F})$ ,
- (ii)  $\mathcal{F}$  is degenerate if and only if there exist  $X \in \Gamma(T\mathcal{F})$  such that  $[\xi, X] \in \Gamma(T\mathcal{F})$ ,
- (iii)  $\mathcal{F}$  is non-degenerate if and only if  $[\xi, X] \notin \Gamma(T\mathcal{F})$  for all  $X \in \Gamma(T\mathcal{F})$ .

An interesting subclass of non-degenerate Legendrian foliations is given by those for which  $\Pi$  is positive definite. Then any such Legendrian foliation is called a *positive definite Legendrian foliation*.

It should be remarked that analogous definitions can be given also for Legendrian distributions. Indeed let  $L$  be a Legendrian distribution and define a bilinear map  $\Pi_L$  on  $L$  by setting  $\Pi_L(X, X') = -(\mathcal{L}_X \mathcal{L}_{X'} \eta)(\xi)$  for all  $X, X' \in \Gamma(L)$ . Then an easy computation yields  $\Pi_L(X, X') = -\eta([\xi, X], X')$ . Next, the condition  $d\eta(X, X') = 0$  implies that  $[X, X'] \in \Gamma(\mathcal{D})$  and hence also  $[[X, X'], \xi] \in \Gamma(\mathcal{D})$ . Thus  $\Pi_L(X, X') = -\eta([\xi, X], X') = \eta([X, X'], \xi) + \eta([X', \xi], X) = -\eta([X', X], \xi) = \Pi_L(X', X)$  and we conclude that  $\Pi_L$  is symmetric. Therefore we can speak of non-degenerate, degenerate and flat Legendrian distributions.

By an *almost bi-Legendrian manifold* we mean a contact manifold  $(M, \eta)$  endowed with two transversal Legendrian distributions  $L_1$  and  $L_2$ . Thus, in particular, the tangent bundle of  $M$  splits up as the direct sum  $TM = L_1 \oplus L_2 \oplus \mathbb{R}\xi$ . When both  $L_1$  and  $L_2$  are integrable we speak of a *bi-Legendrian manifold* ([8]). An (almost) bi-Legendrian manifold is said to be flat, degenerate or non-degenerate if and only if both the Legendrian distributions are flat, degenerate or non-degenerate, respectively.

In [8] to any almost bi-Legendrian manifold it has been attached a canonical connection which plays an important role in the study of almost bi-Legendrian manifolds.

**Theorem 2.4** ([7]). *Let  $(M, \eta, L_1, L_2)$  be an almost bi-Legendrian manifold. There exists a unique connection  $\nabla^{bl}$  such that*

- (i)  $\nabla^{bl} L_1 \subset L_1, \nabla^{bl} L_2 \subset L_2, \nabla^{bl}(\mathbb{R}\xi) \subset \mathbb{R}\xi$ ,
- (ii)  $\nabla^{bl} d\eta = 0$ ,
- (iii)  $T^{bl}(X, Y) = 2d\eta(X, Y)\xi$  for all  $X \in \Gamma(L_1), Y \in \Gamma(L_2)$ ,  
 $T^{bl}(X, \xi) = [\xi, X_{L_1}]_{L_2} + [\xi, X_{L_2}]_{L_1}$  for all  $X \in \Gamma(TM)$ ,

where  $T^{bl}$  denotes the torsion tensor of  $\nabla^{bl}$  and  $X_{L_1}$  and  $X_{L_2}$  the projections of  $X$  onto the subbundles  $L_1$  and  $L_2$  of  $TM$ , respectively, according to the decomposition  $TM = L_1 \oplus L_2 \oplus \mathbb{R}\xi$ .

Such a connection is called the *bi-Legendrian connection* of the almost bi-Legendrian manifold  $(M, \eta, L_1, L_2)$ . Its explicit definition is the following ([7]). Let  $H : TM \rightarrow TM$  be the operator defined by setting, for all  $Z, Z' \in \Gamma(TM)$ ,  $H(Z, Z')$  the unique section of  $\mathcal{D}$  satisfying  $i_{H(Z, Z')} d\eta|_{\mathcal{D}} = (\mathcal{L}_Z i_{Z'} d\eta)|_{\mathcal{D}}$ . Then we set  $\nabla^{bl}\xi := 0$  and, for any  $X \in \Gamma(L_1), Y \in \Gamma(L_2), Z \in \Gamma(TM)$ ,

$$\begin{aligned}\nabla_Z^{bl} X &:= H(Z_{L_1}, X)_{L_1} + [Z_{L_2}, X]_{L_1} + [Z_{\mathbb{R}\xi}, X]_{L_1}, \\ \nabla_Z^{bl} Y &:= H(Z_{L_2}, Y)_{L_2} + [Z_{L_1}, Y]_{L_2} + [Z_{\mathbb{R}\xi}, Y]_{L_2}.\end{aligned}$$

Further properties of that connection are collected in the following proposition.

**Proposition 2.5** ([8]). *Let  $(M, \eta, L_1, L_2)$  be an almost bi-Legendrian manifold and let  $\nabla^{bl}$  denote the corresponding bi-Legendrian connection. Then the 1-form  $\eta$  is  $\nabla^{bl}$ -parallel and the torsion tensor field is given by  $T^{bl}(X, X') = -[X, X']_{L_2}$  for all  $X, X' \in \Gamma(L_1)$  and  $T^{bl}(Y, Y') = -[Y, Y']_{L_1}$  for all  $Y, Y' \in \Gamma(L_2)$ . Moreover, if  $L_1, L_2$  are integrable and flat, the curvature tensor field of  $\nabla^{bl}$  vanishes along the leaves of the foliations defined by  $L_1$ ,  $L_1 \oplus \mathbb{R}\xi$ ,  $L_2$  and  $L_2 \oplus \mathbb{R}\xi$ .*

**2.2. Paracontact manifolds.** A  $(2n+1)$ -dimensional smooth manifold  $M$  has an *almost paracontact structure* ([18]) if it admits a  $(1, 1)$ -tensor field  $\psi$ , a vector field  $\xi$  and a 1-form  $\eta$  satisfying the following conditions

- (i)  $\eta(\xi) = 1$ ,  $\psi^2 = I - \eta \otimes \xi$ ,
- (ii) denoted by  $\mathcal{D}$  the  $2n$ -dimensional distribution generated by  $\eta$ , the tensor field  $\psi$  induces an almost paracomplex structure on each fibre on  $\mathcal{D}$ .

Recall that an almost paracomplex structure on a  $2n$ -dimensional smooth manifold is a tensor field  $J$  of type  $(1, 1)$  such that  $J \neq I$ ,  $J^2 = I$  and the eigendistributions  $T^+, T^-$  corresponding to the eigenvalues  $1, -1$  of  $J$ , respectively, have equal dimension  $n$ .

As an immediate consequence of the definition one has that  $\psi\xi = 0$ ,  $\eta \circ \psi = 0$  and the field of endomorphisms  $\psi$  has constant rank  $2n$ . Any almost paracontact manifold admits a semi-Riemannian metric  $g$  such that

$$(2.2) \quad g(\psi X, \psi Y) = -g(X, Y) + \eta(X)\eta(Y)$$

for all  $X, Y \in \Gamma(TM)$ . Then  $(M, \psi, \xi, \eta, g)$  is called an *almost paracontact metric manifold*. Note that any such semi-Riemannian metric is necessarily of signature  $(n+1, n)$ . If in addition  $d\eta(X, Y) = g(X, \psi Y)$  for all  $X, Y \in \Gamma(TM)$ ,  $(M, \psi, \xi, \eta, g)$  is said to be a *paracontact metric manifold*.

On an almost paracontact manifold one defines the tensor field  $N^{(1)} = N_\psi - d\eta \otimes \xi$ , where  $N_\psi$  is the Nijenhuis tensor of  $\psi$ , defined as  $N_\psi(X, Y) = \psi^2[X, Y] + [\psi X, \psi Y] - \psi[\psi X, Y] - \psi[X, \psi Y]$ . If  $N^{(1)}$  vanishes identically the almost paracontact manifold is said to be *normal*. We prove the following characterization of the normality in terms of foliations.

**Proposition 2.6.** *Let  $(M, \psi, \xi, \eta)$  be an almost paracontact manifold. Let  $T^+$  and  $T^-$  be the eigendistributions of  $\psi|_{\mathcal{D}}$  corresponding to the eigenvalues  $1, -1$ , respectively. Then  $M$  is normal if and only if  $T^+$  and  $T^-$  are involutive and  $\xi$  is foliate with respect to both  $T^+$  and  $T^-$ .*

*Proof.* Assume that  $N^{(1)}$  vanishes identically. In particular, for any  $X, Y \in \Gamma(T^+)$  we have

$$\begin{aligned} 0 &= [X, Y] - \eta([X, Y])\xi + [X, Y] - \psi[X, Y] - \psi[X, Y] - 2d\eta(X, Y)\xi \\ &= 2[X, Y] - 2\psi[X, Y], \end{aligned}$$

so that the integrability of  $T^+$  follows. Moreover, for any  $X \in \Gamma(T^+)$

$$0 = N^{(1)}(X, \xi) = \psi^2[X, \xi] - \psi[\psi X, \xi] = -\psi[X, \xi] + [X, \xi],$$

and hence  $[X, \xi] \in \Gamma(T^+)$ . In a similar way the analogous assertions for  $T^-$  can be proved. Conversely, assume that  $T^+$  and  $T^-$  are both integrable. Then for all

$X, Y \in \Gamma(T^+)$  we have

$$\begin{aligned} N^{(1)}(X, Y) &= \psi^2[X, Y] + [\psi X, \psi Y] - \psi[\psi X, Y] - \psi[X, \psi Y] - 2d\eta(X, Y)\xi \\ &= [X, Y] + [X, Y] - [X, Y] - [X, Y] = 0 \end{aligned}$$

and, analogously,  $N^{(1)}(X, Y) = 0$  for all  $X, Y \in \Gamma(T^-)$ . Next, if  $X \in \Gamma(T^+)$  and  $Y \in \Gamma(T^-)$ ,

$$\begin{aligned} N^{(1)}(X, Y) &= \psi^2[X, Y] + [\psi X, \psi Y] - \psi[\psi X, Y] - \psi[X, \psi Y] - 2d\eta(X, Y)\xi \\ &= [X, Y] - \eta([X, Y])\xi - [X, Y] - \psi[X, Y] + \psi[X, Y] - 2d\eta(X, Y)\xi = 0. \end{aligned}$$

Finally, for all  $X \in \Gamma(T^+)$  we have  $N^{(1)}(X, \xi) = [X, \xi] - \psi[X, \xi] = 0$ , since  $\xi$  is foliate with respect to  $T^+$ . Similarly one has  $N^{(1)}(Y, \xi) = 0$  for all  $Y \in \Gamma(T^-)$ .  $\square$

In a paracontact metric manifold one defines a symmetric, trace-free operator  $h := \frac{1}{2}\mathcal{L}_\xi\psi$ .  $h$  anti-commutes with  $\psi$  and satisfies  $h\xi = 0$  and  $\nabla\xi = -\psi + \psi h$ . Moreover  $h \equiv 0$  if and only if  $\xi$  is a Killing vector field and in this case  $(M, \psi, \xi, \eta, g)$  is said to be a *K-paracontact manifold*. A normal paracontact metric manifold is called a *para-Sasakian manifold*. We have the following characterization.

**Theorem 2.7** ([22]). *An almost paracontact metric structure  $(\psi, \xi, \eta, g)$  is para-Sasakian if and only if, for all  $X, Y \in \Gamma(TM)$ ,*

$$(2.3) \quad (\nabla_X \psi)Y = -g(X, Y)\xi + \eta(Y)X.$$

*In particular, any para-Sasakian manifold is K-paracontact.*

In any paracontact metric manifold S. Zamkovoy introduced a canonical connection which plays in paracontact geometry the same role of the generalized Tanaka-Webster connection ([21]) in a contact metric manifold.

**Theorem 2.8** ([22]). *On a paracontact metric manifold there exists a unique connection  $\nabla^{pc}$ , called the canonical paracontact connection, satisfying the following properties:*

- (i)  $\nabla^{pc}\eta = 0, \nabla^{pc}\xi = 0, \nabla^{pc}g = 0,$
- (ii)  $(\nabla_X^{pc}\psi)Y = (\nabla_X\psi)Y + g(X - hX, Y)\xi - \eta(Y)(X - hX),$
- (iii)  $T^{pc}(\xi, \psi Y) = -\psi T^{pc}(\xi, Y),$
- (iv)  $T^{pc}(X, Y) = 2d\eta(X, Y)\xi$  on  $\mathcal{D} = \ker(\eta)$ .

*The explicit expression of this connection is given by*

$$(2.4) \quad \nabla_X^{pc}Y = \nabla_X Y + \eta(X)\psi Y + \eta(Y)(\psi X - \psi hX) + g(X - hX, \psi Y)\xi.$$

*Moreover, the tensor torsion field is given by*

$$(2.5) \quad T^{pc}(X, Y) = \eta(X)\psi hY - \eta(Y)\psi hX + 2g(X, \psi Y)\xi.$$

An almost paracontact structure  $(\psi, \xi, \eta)$  is said to be *integrable* ([22]) if the almost paracomplex structure  $\psi|_{\mathcal{D}}$  satisfies the condition  $N_\psi(X, Y) \in \Gamma(\mathbb{R}\xi)$  for all  $X, Y \in \Gamma(\mathcal{D})$ . This is equivalent to require that the eigendistributions  $T^\pm$  of  $\psi$  are formally integrable, in the sense that  $[T^\pm, T^\pm] \subset T^\pm \oplus \mathbb{R}\xi$ . For an integrable paracontact metric manifold, the canonical paracontact connection shares many of the properties of the Tanaka-Webster connection on CR-manifolds. For instance we have the following result.

**Theorem 2.9** ([22]). *A paracontact metric manifold  $(M, \psi, \xi, \eta, g)$  is integrable if and only if the canonical paracontact connection preserves the structure tensor  $\psi$ .*

In particular, from Theorem 2.9 and Theorem 2.7 it follows that any para-Sasakian manifold is integrable.

### 3. THE MAIN RESULTS

In this section we prove that paracontact geometry and the theory of almost bi-Legendrian manifolds are strictly connected. More precisely we have the following result.

**Theorem 3.1.** *There is a biunivocal correspondence between almost bi-Legendrian structures and paracontact metric structures.*

*Proof.* Let  $(M, \eta, L_1, L_2)$  be an almost bi-Legendrian manifold. We define a  $(1, 1)$ -tensor field  $\psi$  on  $M$  by setting  $\psi|_{L_1} = I$ ,  $\psi|_{L_2} = -I$  and  $\psi\xi = 0$ . Moreover we put

$$(3.1) \quad g(X, Y) := d\eta(X, \psi Y) + \eta(X)\eta(Y)$$

for all  $X, Y \in \Gamma(TM)$ . We prove that  $(M, \psi, \xi, \eta, g)$  is in fact a paracontact metric manifold. A straightforward computation shows that  $\psi^2 = I - \eta \otimes \xi$ . Moreover, by construction,  $\psi$  induces an almost paracomplex structure on  $\mathcal{D}$  since  $\psi|_{\mathcal{D}}^2 = I$  and the eigendistributions corresponding to the eigenvalues  $1, -1$  of  $\psi|_{\mathcal{D}}$  are  $L_1$  and  $L_2$ , respectively. It remains to check that  $g$  is a compatible metric such that  $g(X, \psi Y) = d\eta(X, Y)$ . Indeed we have, for any  $X, Y \in \Gamma(TM)$ ,

$$\begin{aligned} g(\psi X, \psi Y) &= d\eta(\psi X, Y - \eta(Y)\xi) + \eta(\psi X)\eta(\psi Y) \\ &= d\eta(\psi X, Y) \\ &= -g(X, Y) + \eta(X)\eta(Y). \end{aligned}$$

Then  $g(X, \psi Y) = d\eta(X, \psi^2 Y) + \eta(X)\eta(\psi Y) = d\eta(X, Y - \eta(Y)\xi) = d\eta(X, Y)$  for all  $X, Y \in \Gamma(TM)$ . Conversely, let  $(M, \psi, \xi, \eta, g)$  be a paracontact metric manifold. Then  $\eta \wedge (d\eta)^n \neq 0$  everywhere on  $M$  ([22]), that is  $\eta$  is a contact form on  $M$ . We define an almost bi-Legendrian structure  $(L_1, L_2)$  on  $M$  by setting  $L_1 := T^+$  and  $L_2 := T^-$ . Note that  $T^+$  and  $T^-$  are  $g$ -isotropic distributions, that is  $g(X, Y) = 0$  for all  $X, Y \in \Gamma(T^\pm)$ . Indeed by (2.2) one has  $g(X, Y) = g(\psi X, \psi Y) = -g(X, Y)$ , so that  $g(X, Y) = 0$ . Consequently  $L_1$  and  $L_2$  are in fact Legendrian distributions on  $M$ , since they are  $n$ -dimensional subbundles of  $\mathcal{D} = \ker(\eta)$  satisfying  $d\eta(X, X') = 0$  for all  $X, X' \in \Gamma(L_1)$ ,  $d\eta(Y, Y') = 0$  for all  $Y, Y' \in \Gamma(L_2)$ , and they are mutually transversal.  $\square$

From now on, we shall identify a paracontact metric manifold with the canonical induced almost bi-Legendrian structure, according to Theorem 3.1. In the next results, we shall investigate how that bijection acts on special classes of paracontact metric manifolds.

**Corollary 3.2.** *There is a biunivocal correspondence between flat almost bi-Legendrian structures and  $K$ -paracontact structures.*

*Proof.* First, we give an explicit expression of the tensor field  $h$  in terms of the bi-Legendrian structure  $(L_1, L_2)$  which is identified with the pair of eigendistributions  $(T^+, T^-)$  of the corresponding paracontact metric structure  $(\psi, \xi, \eta, g)$ , according

to Theorem 3.1. Indeed for any  $X \in \Gamma(L_1)$ , we have

$$\begin{aligned}
 hX &= \frac{1}{2}([\xi, X] - \psi([\xi, X]_{L_1} + [\xi, X]_{L_2})) \\
 (3.2) \quad &= \frac{1}{2}([\xi, X] - [\xi, X]_{L_1} + [\xi, X]_{L_2}) \\
 &= [\xi, X]_{L_2}.
 \end{aligned}$$

Analogously one obtains that  $hY = -[\xi, Y]_{L_1}$  for all  $Y \in \Gamma(L_2)$ . Therefore  $(\psi, \xi, \eta, g)$  is a  $K$ -paracontact structure if and only if both  $L_1$  and  $L_2$  are flat as Legendrian distributions. This remark together with Theorem 3.1 proves the assertion.  $\square$

**Corollary 3.3.** *There is a biunivocal correspondence between bi-Legendrian structures and integrable paracontact metric structures.*

*Proof.* We have only to prove that an integrable paracontact metric structure gives rise to a bi-Legendrian structure. Let  $(M, \psi, \xi, \eta, g)$  be an integrable paracontact metric manifold. According to Theorem 3.1 we define an almost bi-Legendrian structure on  $(M, \eta)$  by setting  $L_1 := T^+$  and  $L_2 := T^-$ . Since  $(\psi, \xi, \eta)$  is integrable we have  $[L_i, L_i] \subset L_i \oplus \mathbb{R}\xi$ ,  $i \in \{1, 2\}$ . But  $L_i$  are Legendrian distributions, so that  $[L_i, L_i] \subset \mathcal{D}$ ,  $i \in \{1, 2\}$ . Hence  $L_1$  and  $L_2$  are necessarily involutive.  $\square$

Finally, the following result, which is a direct consequence of Theorem 3.1 and Proposition 2.6, holds.

**Corollary 3.4.** *There is a biunivocal correspondence between flat bi-Legendrian structures and para-Sasakian structures.*

We recall that, given an almost bi-Legendrian manifold  $(M, \eta, L_1, L_2)$ , an *almost bi-Legendrian equivalence* is a contactomorphism  $f$  of  $M$  which preserves the Lagrangian distributions  $L_1$  and  $L_2$ , that is  $f^*\eta = \eta$ ,  $f_{*x}\xi_x = \xi_{f(x)}$  and  $f_{*x}(L_{1x}) = L_{1f(x)}$ ,  $f_{*x}(L_{2x}) = L_{2f(x)}$  for all  $x \in M$  ([20]). However, in view of Theorem 3.1 this notion reflects in the corresponding paracontact structure as follows.

**Proposition 3.5.** *Let  $(M, \eta, L_1, L_2)$  be an almost bi-Legendrian manifold. Then a diffeomorphism  $f : M \rightarrow M$  is an almost bi-Legendrian equivalence if and only if it is an automorphism with respect to the induced paracontact metric structure.*

*Proof.* We recall that an automorphism of a paracontact metric manifold  $(M, \psi, \xi, \eta, g)$  is nothing but an isometry  $f : M \rightarrow M$  such that  $\psi \circ f_* = f_* \circ \psi$  and  $f^*\eta = \eta$ . Now let  $f$  be an almost bi-Legendrian equivalence. Then, by the identification  $L_1 \equiv T^+$ ,  $L_2 \equiv T^-$  one has, for all  $X \in \Gamma(L_1)$  and  $Y \in \Gamma(L_2)$ ,  $\psi f_*X = f_*X = f_*\psi X$  and  $\psi f_*Y = -f_*Y = f_*\psi Y$ , so that, since  $f_*\xi = \xi$ , it follows that  $\psi \circ f_* = f_* \circ \psi$ . Then, for all  $X, Y \in \Gamma(TM)$ ,  $g(f_*X, f_*Y) = d\eta(f_*X, \psi f_*Y) = d\eta(f_*X, f_*\psi Y) = (f^*d\eta)(X, \psi Y) = d\eta(X, \psi Y) = g(X, Y)$ . Conversely, if  $f$  is an automorphism of the paracontact metric manifold  $(M, \psi, \xi, \eta, g)$ , then for any  $X \in \Gamma(L_1)$  we have  $\psi f_*X = f_*\psi X = f_*X$  and, consequently,  $f_*X \in \Gamma(L_1)$ . Analogously one can prove that  $f$  preserves the Legendrian distribution  $L_2$ .  $\square$

In particular, Theorem 3.1 and Proposition 3.5 permits to provide a classification of paracontact metric structures in the following way. Let  $(\psi, \xi, \eta, g)$  be a paracontact metric structure on the contact manifold  $(M, \eta)$ . By Theorem 3.1 we

can associate with  $(\psi, \xi, \eta, g)$  a canonical almost bi-Legendrian structure  $(L_1, L_2)$ . Hence we can consider the bilinear forms  $\Pi_{L_1}$  and  $\Pi_{L_2}$  of each Legendrian distribution, according to § 2.1. The bilinear forms  $\Pi_{L_1}$  and  $\Pi_{L_2}$  are invariant under almost bi-Legendrian equivalences and hence, by Proposition 3.5, also under automorphisms of the paracontact metric structure  $(\psi, \xi, \eta, g)$ . Therefore  $\Pi_{L_1}$  and  $\Pi_{L_2}$  may be considered as invariants of the given paracontact metric structure  $(\psi, \xi, \eta, g)$  and they can be easily used for classifying paracontact metric structures according to the flatness, degeneracy or non-degeneracy of  $\Pi_{L_1}$  and  $\Pi_{L_2}$ . A further criterion is the integrability/non-integrability of each of the Legendrian distributions  $L_1$  and  $L_2$ . For instance, we have the class given by those paracontact metric structures such that the induced almost bi-Legendrian structure is integrable and flat. By Corollary 3.4 this class corresponds to para-Sasakian structures. The total number of classes amounts to 36.

The study of the interplays between paracontact and bi-Legendrian manifolds is also motivated by the following theorem, whose consequences will be discussed in the sequel.

**Theorem 3.6.** *Let  $(M, \eta, L_1, L_2)$  be an almost bi-Legendrian manifold and let  $(\psi, \xi, \eta, g)$  be the canonical paracontact metric structure induced on  $M$ , according to Theorem 3.1. Let  $\nabla^{bl}$  and  $\nabla^{pc}$  be the corresponding bi-Legendrian and canonical paracontact connections. Then*

- (a)  $\nabla^{bl}\psi = 0$ ,  $\nabla^{bl}g = 0$ ;
- (b) *the bi-Legendrian and the canonical paracontact connections coincide if and only if the induced paracontact metric structure is integrable.*

*Proof.* (a) For any  $V \in \Gamma(TM)$ ,  $X \in \Gamma(L_1)$ ,  $Y \in \Gamma(L_2)$  one has

$$(\nabla_V^{bl}\psi)X = \nabla_V^{bl}\psi X - \psi \nabla_V^{bl}X = \nabla_V^{bl}X - \nabla_V^{bl}X = 0$$

and

$$(\nabla_V^{bl}\psi)Y = \nabla_V^{bl}\psi Y - \psi \nabla_V^{bl}Y = -\nabla_V^{bl}Y + \nabla_V^{bl}Y = 0,$$

since  $\nabla^{bl}$  preserves  $L_1$  and  $L_2$ . Moreover,  $(\nabla_V^{bl}\psi)\xi = \nabla_V^{bl}\psi\xi - \psi \nabla_V^{bl}\xi = 0$ , since  $\nabla^{BC}\xi = 0$ . Then, from the fact that  $\nabla^{bl}\psi = 0$ ,  $\nabla^{bl}\eta = 0$  and  $\nabla^{bl}d\eta = 0$ , we have for all  $X, Y, Z \in \Gamma(TM)$

$$\begin{aligned} (\nabla_X^{bl}g)(Y, Z) &= X(d\eta(Y, \psi Z) + \eta(Y)\eta(Z)) - d\eta(\nabla_X^{bl}Y, \psi Z) - d\eta(Y, \psi \nabla_X^{bl}Z) \\ &= (\nabla_X^{bl}d\eta)(Y, \psi Z) + (\nabla_X^{bl}\eta)(Y)\eta(Z) + \eta(Y)(\nabla_X^{bl}\eta)(Z) = 0. \end{aligned}$$

(b) We have to check that the bi-Legendrian connection of  $(M, \eta, L_1, L_2)$  satisfies the statements (i)–(v) of Theorem 2.8. By Proposition 2.5 and the definition of  $\nabla^{bl}$  we have  $\nabla^{bl}\xi = \nabla^{bl}\eta = 0$  and, by (a),  $\nabla^{bl}g = 0$ . Taking the properties of the torsion tensor field, (iii) of Theorem 2.4 and (iv) of Theorem 2.8 into account, we have that  $T^{bl}(X, Y) = 2d\eta(X, Y)\xi = T^{pc}(X, Y)$  for all  $X \in \Gamma(L_1)$  and  $Y \in \Gamma(L_2)$ , and, by (2.5), (3.2),

$$T^{pc}(X, \xi) = -\psi hX = h\psi X = hX = [\xi, X]_{L_2} = T^{bl}(X, \xi),$$

$$T^{pc}(Y, \xi) = -\psi hY = h\psi Y = -hY = [\xi, Y]_{L_1} = T^{bl}(Y, \xi).$$

Next, by Proposition 2.5 we have that, for any  $X, X' \in \Gamma(L_1)$ ,  $Y, Y' \in \Gamma(L_2)$ ,  $T^{bl}(X, X') = -[X, X']_{L_2}$ ,  $T^{bl}(Y, Y') = -[Y, Y']_{L_1}$ , and  $T^{pc}(X, X') = 2d\eta(X, X')\xi = 0$ ,  $T^{pc}(Y, Y') = 2d\eta(Y, Y')\xi = 0$ , so that the torsion tensor fields

of the two connections  $\nabla^{bl}$  and  $\nabla^{pc}$  coincide if and only if the Legendrian distributions  $L_1$  and  $L_2$  are involutive. In view of Corollary 3.3 this is equivalent to the integrability of the induced paracontact metric structure. Finally, by virtue of (a), the bi-Legendrian connection satisfies (ii) of Theorem 2.8 if and only if  $\nabla^{pc}\psi = 0$ . But, by Theorem 2.9, that last condition is equivalent to the integrability of the paracontact metric structure. The theorem is thus completely proved.  $\square$

**Corollary 3.7.** *Any bi-Legendrian manifold  $(M, \eta, \mathcal{F}_1, \mathcal{F}_2)$  admits a canonical paracontact metric structure whose paracontact connection coincides with the bi-Legendrian connection of  $(M, \eta, \mathcal{F}_1, \mathcal{F}_2)$ .*

*Remark 3.8.* Note that the explicit definition of the bi-Legendrian connection, which we have recalled in § 2.1, is rather involved. Thus Corollary 3.7 clarifies its nature: the bi-Legendrian connection can be seen as the canonical paracontact connection of an integrable paracontact metric structure. Moreover, by (2.4), we can also deduce a relation between the bi-Legendrian connection and the Levi-Civita connection.

Theorem 3.1 and its corollaries yield some consequences for the theory of paracontact manifolds, especially for para-Sasakian geometry. Indeed many of the known results about bi-Legendrian manifolds may be transported to the induced paracontact metric structure. A crucial role in such matters is played by the relations between the bi-Legendrian and the canonical paracontact connection proved in Theorem 3.6.

We recall that the bi-Legendrian connection of an almost bi-Legendrian manifold  $(M, \eta, L_1, L_2)$  is said to be *tangential* ([8]) if its curvature tensor field satisfies  $R^{bl}(X, Y) = 0$  for all  $X \in \Gamma(L_1)$  and  $Y \in \Gamma(L_2)$ . The geometric meaning of tangentiality is explained in [8], where some strong consequences on the geometry of the manifold are proved. Then we have the following results.

**Proposition 3.9.** *Let  $(M, \psi, \xi, \eta, g)$  be a  $(2n+1)$ -dimensional  $K$ -paracontact manifold. Suppose that one among  $T^+$  and  $T^-$  is integrable and the associated bi-Legendrian connection is tangential. Then  $\text{Pont}^j(TM)$  vanishes for  $j > n$ , where  $\text{Pont}(TM)$  denotes the Pontryagin algebra of the bundle  $TM$ .*

*Proof.* The assertion follows by applying [8, Theorem 5.4] to the flat almost bi-Legendrian structure induced on  $M$  by the  $K$ -paracontact structure  $(\psi, \xi, \eta, g)$  according to Corollary 3.2.  $\square$

**Proposition 3.10.** *Let  $(M, \psi, \xi, \eta, g)$  be a compact connected  $K$ -paracontact manifold. Suppose that  $T^+$  (respectively,  $T^-$ ) is integrable and the corresponding bi-Legendrian connection is tangential. If the leaves of the foliation defined by  $T^+$  (respectively,  $T^-$ ) are complete affine manifolds (with respect to the bi-Legendrian connection), then the subbundle  $T^- \oplus \mathbb{R}\xi$  (respectively,  $T^+ \oplus \mathbb{R}\xi$ ) is an Ehresmann connection, in the sense of [6], for the foliation defined by  $T^+$  (respectively,  $T^-$ ).*

*Proof.* The assertion follows from [8, Theorem 6.2].  $\square$

**Corollary 3.11.** *Let  $(M, \psi, \xi, \eta, g)$  be a compact connected para-Sasakian manifold. Assume that the canonical paracontact connection  $\nabla^{pc}$  is tangential and the leaves of the foliation defined by  $T^+$  (respectively,  $T^-$ ) are complete affine manifolds with respect to  $\nabla^{pc}$ . Then the subbundle  $D^- = T^- \oplus \mathbb{R}\xi$  (respectively,  $D^+ = T^+ \oplus \mathbb{R}\xi$ )*

is an Ehresmann connection for the foliation defined by  $T^+$  (respectively,  $T^-$ ). Furthermore, the universal covers of any two leaves of  $T^+$  (respectively,  $T^-$ ) are isomorphic and the universal cover  $\tilde{M}$  of  $M$  is topologically a product  $\tilde{\mathcal{L}}^+ \times \tilde{\mathcal{D}}^-$  (respectively,  $\tilde{\mathcal{L}}^- \times \tilde{\mathcal{D}}^+$ ), where  $\tilde{\mathcal{L}}^+$  (respectively,  $\tilde{\mathcal{L}}^-$ ) is the universal cover of the leaves of  $T^+$  (respectively,  $T^-$ ) and  $\tilde{\mathcal{D}}^-$  (respectively,  $\tilde{\mathcal{D}}^+$ ) is the universal cover of the leaves of the foliation defined by  $D^-$  (respectively,  $D^+$ ).

*Proof.* The proof follows from Proposition 3.10 and Corollary 6.7, 6.9 in [8], taking into account that  $T^+$  and  $T^-$  are involutive and flat and, by Theorem 3.6, the bi-Legendrian and the canonical paracontact connections coincide.  $\square$

**Proposition 3.12.** *Let  $(M, \psi, \xi, \eta, g)$  be a para-Sasakian manifold. Then the canonical paracontact connection is flat along the leaves of the foliations defined by the eigendistributions  $T^+$  and  $T^-$ .*

*Proof.* Since  $M$  is para-Sasakian, the paracontact structure  $(\psi, \xi, \eta)$  is normal, in particular integrable. Hence by Theorem 3.6 the canonical paracontact connection  $\nabla^{pc}$  coincides with the bi-Legendrian connection  $\nabla^{bl}$  of the induced flat bi-Legendrian structure defined by  $(T^+, T^-)$ , according to Corollary 3.4. Then by applying Proposition 2.5 we get the result.  $\square$

**Corollary 3.13.** *Let  $(M, \psi, \xi, \eta, g)$  be a para-Sasakian manifold. Then the leaves of the foliations defined by the eigendistributions  $T^+$  and  $T^-$  and the leaves of the foliations defined by  $T^+ \oplus \mathbb{R}\xi$  and  $T^- \oplus \mathbb{R}\xi$  admit a canonical (flat) affine structure.*

*Proof.* The first part is a direct consequence of Proposition 3.12. We prove that the leaves of the foliations defined by  $T^+ \oplus \mathbb{R}\xi$  and  $T^- \oplus \mathbb{R}\xi$  admit a canonical (flat) affine structure. Indeed, by (2.5) the torsion of the canonical paracontact connection satisfies  $T^{pc}(Z, Z') = 2d\eta(Z, Z') = -\eta([Z, Z']) = 0$  for all  $Z, Z' \in \Gamma(T^\pm)$  and  $T^{pc}(Z, \xi) = \pm hZ = 0$  for all  $Z \in \Gamma(T^\pm)$ , since the distributions  $T^+$  and  $T^-$  are integrable and  $h = 0$ ,  $M$  being para-Sasakian. Moreover by Theorem 3.6  $\nabla^{pc} = \nabla^{bl}$ , hence using Proposition 2.5, we have that  $R^{pc}(Z, \xi) = 0$  for all  $Z \in \Gamma(T^\pm)$ . Thus the connection induced by the canonical paracontact connection on the leaves of the foliations defined by  $T^+ \oplus \mathbb{R}\xi$  and  $T^- \oplus \mathbb{R}\xi$  provides the desired flat affine structure.  $\square$

#### 4. EXAMPLES AND REMARKS

In this section we shall present some wide classes of examples of bi-Legendrian manifolds and thus, in turn, again by Theorem 3.1, of paracontact metric structures. Moreover we prove a theorem of local equivalence for  $\nabla^{pc}$ -flat paracontact manifolds. Let us begin with a method for constructing new almost bi-Legendrian structures from a given Legendrian distribution.

**4.1. Conjugate Legendrian foliation.** Let  $L$  be a Legendrian distribution on a contact manifold  $(M, \eta)$ . We show that there exists at least one Legendrian distribution on  $M$  transversal to  $L$ . Indeed, since  $(M, \eta)$  is a contact manifold it admits a Riemannian metric  $G$  and a  $(1, 1)$ -tensor field  $\phi$  satisfying

$$(4.1) \quad \phi^2 = -I + \eta \otimes \xi, G(\phi X, \phi Y) = G(X, Y) - \eta(X)\eta(Y), G(X, \phi Y) = d\eta(X, Y)$$

for all  $X, Y \in \Gamma(TM)$  (cf. [5]).  $G$  is in fact an associated metric (§ 2.1). Note that from (4.1) it follows that

$$(4.2) \quad G(X, Y) = -d\eta(X, \phi Y) + \eta(X)\eta(Y).$$

Let  $Q$  be the distribution defined by  $Q := \phi L = \mathcal{D} \cap L^\perp$ . As a matter of fact  $Q$  is a Legendrian distribution on  $M$  called the *conjugate Legendrian distribution* of  $L$  with respect to the contact metric structure  $(\phi, \xi, \eta, G)$  ([17]). Thus  $(L, Q)$  defines a  $G$ -orthogonal almost bi-Legendrian structure on  $M$ . It should be remarked that under the assumption that  $\xi$  is Killing, if  $L$  is flat, degenerate or non-degenerate then also its conjugate is flat, degenerate or non-degenerate, respectively ([17]).

**Proposition 4.1.** *Let  $L$  be a Legendrian distribution on a contact manifold  $(M, \eta)$  and let  $Q$  be its conjugate Legendrian distribution. Then starting from  $(L, Q)$  one can define infinitely many almost bi-Legendrian structures on  $(M, \eta)$ .*

*Proof.* Let  $(\psi, \xi, \eta, g)$  be the paracontact metric structure associated with the almost bi-Legendrian structure  $(L, Q)$ . Note that

$$(4.3) \quad \phi \circ \psi = -\psi \circ \phi.$$

Indeed, for any  $X \in \Gamma(L) = \Gamma(T^+)$ ,  $\phi\psi X = \phi X = -\psi\phi X$  and, for any  $Y \in \Gamma(Q) = \Gamma(T^-)$ ,  $\phi\psi Y = -\phi Y = -\psi\phi Y$ , since  $\phi L = Q$  and  $\phi Q = L$ . Finally,  $\phi\psi\xi = 0 = -\psi\phi\xi$ . Now, let  $\alpha, \beta \in \mathbb{R}$  such that  $\alpha^2 + \beta^2 = 1$  and let  $\psi_{\alpha, \beta}$  be the tensor field of type  $(1, 1)$  defined by  $\psi_{\alpha, \beta} = \alpha\psi + \beta\phi \circ \psi$ . One has, for any  $X \in \Gamma(TM)$ ,

$$\begin{aligned} \psi_{\alpha, \beta}^2 X &= \alpha^2 \psi^2 X + \alpha\beta\psi\phi\psi X + \beta\alpha\phi\psi^2 X + \beta^2\phi\psi\phi\psi X \\ &= \alpha^2 X - \alpha^2 \eta(X)\xi - \alpha\beta\phi X + \alpha\beta\phi X + \beta^2 X - \beta^2 \eta(X)\xi \\ &= X - \eta(X)\xi. \end{aligned}$$

Thus  $(\psi_{\alpha, \beta}, \xi, \eta)$  is an almost paracontact structure on  $M$  and the eigenspace distributions  $T_{\alpha, \beta}^+$  and  $T_{\alpha, \beta}^-$  of  $\psi_{\alpha, \beta}|_{\mathcal{D}}$  define an almost bi-Legendrian structure on  $M$ . Indeed, let  $X$  and  $X'$  be two sections of  $T_{\alpha, \beta}^+$ . So  $X = \psi_{\alpha, \beta} X = \alpha\psi X + \beta\phi\psi X$  and  $X' = \psi_{\alpha, \beta} X' = \alpha\psi X' + \beta\phi\psi X'$ . Then we have

$$\begin{aligned} d\eta(X, X') &= \alpha^2 d\eta(\psi X, \psi X') + \alpha\beta d\eta(\psi X, \phi\psi X') + \beta\alpha d\eta(\phi\psi X, \psi X') \\ &\quad + \beta^2 d\eta(\phi\psi X, \phi\psi X') \\ &= -\alpha^2 d\eta(X, X') + \alpha\beta d\eta(X, \phi X') + \alpha\beta d\eta(\phi X, X') - \beta^2 d\eta(X, X') \\ &= -d\eta(X, X'), \end{aligned}$$

so that  $d\eta(X, X') = 0$ . Analogously one can prove that  $d\eta(Y, Y') = 0$  for all  $Y, Y' \in \Gamma(T_{\alpha, \beta}^-)$ .  $\square$

An interesting case occurs when  $(M, \phi, \xi, \eta, G)$  is a Sasakian manifold. We have in fact the following result.

**Theorem 4.2.** *Let  $(M, \phi, \xi, \eta, G)$  be a Sasakian manifold endowed with a flat Legendrian distribution  $L$ . Let  $Q = \phi L$  be its conjugate Legendrian distribution. Suppose that the generalized Tanaka-Webster connection  $\nabla^{TW}$  preserves the distribution  $L$ . Then we have:*

- (i)  *$L$  and  $Q$  are integrable and  $\nabla^{TW}$  coincides with the bi-Legendrian connection  $\nabla^{bl}$  corresponding to the bi-Legendrian structure  $(L, Q)$ .*

- (ii) *The induced paracontact metric structure  $(\psi, \xi, \eta, g)$  is para-Sasakian and the distributions  $T_{\alpha, \beta}^+$  and  $T_{\alpha, \beta}^-$  are integrable.*
- (iii) *The canonical paracontact connection of  $(\psi, \xi, \eta, g)$  coincides with the generalized Tanaka-Webster connection of  $(\phi, \xi, \eta, G)$ .*

*Proof.* The first part of the statement has been proved in [9]. Next, that the paracontact metric structure  $(\psi, \xi, \eta, g)$  is para-Sasakian follows from Corollary 3.4, taking into account that  $L$  and  $Q$  are integrable and flat. We prove that the Legendrian distributions  $T_{\alpha, \beta}^\pm$ , corresponding to the almost paracontact structures  $(\psi_{\alpha, \beta}, \xi, \eta)$  defined above are involutive. For any  $X, X' \in \Gamma(T_{\alpha, \beta}^+)$  we have, by (2.1) and (2.3),

$$\begin{aligned}
\psi_{\alpha, \beta}[X, X'] &= \alpha\psi\nabla_X X' + \beta\phi\psi\nabla_X X' - \alpha\psi\nabla_{X'} X - \beta\phi\psi\nabla_{X'} X \\
&= -\alpha(\nabla_X \psi)X' + \alpha\nabla_X \psi X' - \beta\phi(\nabla_X \psi)X' + \beta\phi\nabla_X \psi X' \\
&\quad + \alpha(\nabla_{X'} \psi)X - \alpha\nabla_{X'} \psi X + \beta\phi(\nabla_{X'} \psi)X - \beta\phi\nabla_{X'} \psi X \\
&= \alpha g(X, X')\xi - \alpha\eta(X')X + \alpha\nabla_X \psi X' + \beta g(X, X')\phi\xi - \beta\eta(X')\phi X \\
&\quad - \beta(\nabla_X \phi)\psi X' + \beta\nabla_X \phi\psi X' - \alpha g(X, X')\xi + \alpha\eta(X)X' - \alpha\nabla_{X'} \psi X \\
&\quad - \beta g(X, X')\phi\xi + \beta\eta(X)\phi X' + \beta(\nabla_{X'} \phi)\psi X - \beta\nabla_{X'} \phi\psi X \\
&= \nabla_X \psi_{\alpha, \beta} X' - \beta g(X, \psi X')\xi + \beta\eta(\psi X')X - \nabla_{X'} \psi_{\alpha, \beta} X \\
&\quad + \beta g(X', \psi X)\xi - \beta\eta(\psi X)X' \\
&= \nabla_X X' - \nabla_{X'} X - 2\beta d\eta(X, X')\xi \\
&= [X, X']
\end{aligned}$$

since  $d\eta(X, X') = 0$ ,  $T_{\alpha, \beta}^+$  being a Legendrian distribution. By a similar argument one can prove also the integrability of  $T_{\alpha, \beta}^-$ . Thus (ii) is proved. Finally, since  $(\psi, \xi, \eta, g)$  is para-Sasakian, by Theorem 3.6 the canonical paracontact connection coincides with the bi-Legendrian connection of  $(M, \eta, L, Q)$ , which in turn, by (i), coincides with the generalized Tanaka-Webster connection of the Sasakian structure  $(\phi, \xi, \eta, G)$ .  $\square$

**4.2. Unit cotangent bundles.** Another way for attaching to a given Legendrian distribution  $L$  a transversal Legendrian distribution is proposed, by an intrinsic construction, in [20] and [19] under the assumption of the integrability and non-degeneracy of  $L$ . So let  $\mathcal{F}$  be a non-degenerate Legendrian foliation. One defines an operator  $S_{\mathcal{F}} : \mathcal{D} \rightarrow \mathcal{D}$  by setting  $S_{\mathcal{F}} = \frac{1}{2}(i_{\mathcal{D}} - \mathcal{L}_{\xi}\lambda)$ , where  $\lambda$  is the tensor field of type  $(1, 1)$  on  $M$  such that  $\Pi(\lambda Z, X) = d\eta(Z, X)$  for all  $Z \in \Gamma(TM)$  and  $X \in \Gamma(T\mathcal{F})$ . Then the image of  $S_{\mathcal{F}}$  is a Legendrian distribution of  $(M, \eta)$  transversal to  $\mathcal{F}$ . In particular, that construction applies to unit cotangent bundles. Let  $M$  be a  $(n+1)$ -dimensional smooth manifold with local coordinates  $(x_1, \dots, x_{n+1})$ . Then  $q_i = x_i \circ \pi$  and  $p_i$ ,  $i \in \{1, \dots, n+1\}$ , are coordinates on the cotangent bundle  $T^*M$ , where  $(p_1, \dots, p_{n+1})$  are fiber coordinates and  $\pi : T^*M \rightarrow M$  denotes the projection map. The Liouville form on  $T^*M$  is then defined in coordinates by  $\beta = \sum_{i=1}^{n+1} p_i dq_i$ . Now let  $F : T^*M \rightarrow [0, +\infty[$  be a function such that  $F(tv) = tF(v)$  for all  $t \geq 0$  and  $v \in T^*M$ . Then the set  $S_F^*M = \{v \in T^*M | F(v) = 1\}$  is a  $(2n+1)$ -dimensional submanifold of  $T^*M$  called *unit cotangent bundle*. The Liouville form  $\beta$  on  $T^*M$  pulls-back to a contact form  $\eta_F$  on  $S_F^*M$  and the connected components of the fibers of the projection  $\pi : S_F^*M \rightarrow M$  define a Legendrian foliation  $\mathcal{F}_F$  on

$S_F^*M$ . The computation in coordinates of the invariant  $\Pi_{\mathcal{F}_F}$  yields that it is the restriction to the tangent bundle of  $\mathcal{F}_F$  of the symmetric form

$$\Pi_{\mathcal{F}_F} = \sum_{i,j=1}^{n+1} \frac{\partial^2 F}{\partial p_i \partial p_j} dp_i \otimes dp_j.$$

So if the Hessian matrix  $\left\{ \frac{\partial^2 F}{\partial p_i \partial p_j} \right\}$  is not singular then  $\mathcal{F}_F$  defines a non-degenerate Legendrian foliation on  $S_F^*M$ . Thus with the notation above we set  $Q_F := \text{Im}(S_{\mathcal{F}_F})$  defining in this way a canonical almost bi-Legendrian structure on the unit cotangent bundle  $S_F^*M$ . For example, if  $F$  is the norm defined by a Riemannian metric  $g$  on  $M$ , then  $S_F^*M$  is the cotangent sphere bundle  $S_g^*M = T_1^*M$  on  $M$  and the Legendrian foliation  $\mathcal{F}_g$  on  $T_1^*M$  is clearly non-degenerate. By a result of Pang ([20, Proposition 5.30]) any positive definite Legendrian foliation is locally equivalent to one of the form  $\mathcal{F}_F$  with  $F$  a Finslerian metric on  $M$ . In particular this implies the following theorem.

**Proposition 4.3.** *Let  $\mathcal{F}$  be a positive definite Legendrian foliation on a contact manifold  $(M, \eta)$ . Then the almost bi-Legendrian manifold  $(M, \eta, T\mathcal{F}, Q_{\mathcal{F}})$ , where  $Q_{\mathcal{F}} = \text{Im}(S_{\mathcal{F}})$ , is locally equivalent to the almost bi-Legendrian manifold  $(S_F^*M, \eta_F, \mathcal{F}_F, Q_F)$  for some Finslerian metric  $F$  on  $M$ .*

**4.3. Standard bi-Legendrian structure on  $\mathbb{R}^{2n+1}$ .** A standard example of paracontact metric manifold is the following. Consider in  $\mathbb{R}^{2n+1}$  with global coordinates  $x_1, \dots, x_n, y_1, \dots, y_n, z$ , the  $(1,1)$ -tensor field  $\psi$  represented by the matrix

$$\begin{pmatrix} -I_n & 0 & 0 \\ 0 & I_n & 0 \\ -y_i & 0 & 0 \end{pmatrix}$$

and put  $\eta = dz - \sum_{i=1}^n y_i dx_i$  and  $\xi = \frac{\partial}{\partial z}$ . A straightforward computation shows that  $(\psi, \xi, \eta)$  defines an almost paracontact structure on  $\mathbb{R}^{2n+1}$ . Moreover, one can consider the semi-Riemannian metric  $g$  given by

$$\begin{pmatrix} y_i y_j & \frac{1}{2} \delta_{ij} & -y_i \\ \frac{1}{2} \delta_{ij} & 0 & 0 \\ -y_j & 0 & 1 \end{pmatrix}$$

and check that  $g$  is a compatible metric and  $(\mathbb{R}^{2n+1}, \psi, \xi, \eta, g)$  is in fact a para-Sasakian manifold. The corresponding bi-Legendrian structure on  $(\mathbb{R}^{2n+1}, \eta)$  is given by the integrable Legendrian distributions spanned by  $\{X_1, \dots, X_n\}$  and by  $\{Y_1, \dots, Y_n\}$ , where, for each  $i \in \{1, \dots, n\}$ ,  $X_i := \frac{\partial}{\partial y_i}$  and  $Y_i := \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}$ . It is called the *standard bi-Legendrian structure* on  $\mathbb{R}^{2n+1}$  (cf. [7]). One can verify that the canonical paracontact connection is flat. Now we prove that, in a certain sense, also the converse holds.

**Proposition 4.4.** *Let  $(M, \psi, \xi, \eta, g)$  be a para-Sasakian manifold. If the canonical paracontact connection is everywhere flat, then  $(M, \psi, \xi, \eta, g)$  is locally isomorphic to the standard paracontact metric structure of  $\mathbb{R}^{2n+1}$ .*

*Proof.* Let us suppose that the canonical paracontact connection  $\nabla^{pc}$  of  $(M, \psi, \xi, \eta, g)$  is flat. By Theorem 3.6  $\nabla^{pc}$  coincides with the bi-Legendrian connection associated with the flat bi-Legendrian structure defined by  $(T^+, T^-)$ . Thus, by applying [7, Theorem 4.2], we have that this bi-Legendrian structure is locally

equivalent to the standard bi-Legendrian structure on  $\mathbb{R}^{2n+1}$ . Now the assertion follows directly from Proposition 3.5.  $\square$

**4.4. Anosov flows.** We recall for the convenience of the reader some definitions. Let  $M$  be a compact differentiable manifold. The flow  $\{\omega_t\}$  of a non-vanishing vector field  $\xi$  on  $M$  is said to be an *Anosov flow* (or  $\xi$  to be an *Anosov vector field*) if there exist subbundles  $E^s$  and  $E^u$  which are invariant along the flow and such that  $TM = E^s \oplus E^u \oplus \mathbb{R}\xi$  and there exists a Riemannian metric such that

$$(4.4) \quad \begin{aligned} |\omega_{t*}v| &\leq a \exp(-ct)|v| \quad \text{for all } t \geq 0 \text{ and } v \in E_x^s, \\ |\omega_{t*}w| &\leq a \exp(ct)|w| \quad \text{for all } t \leq 0 \text{ and } w \in E_x^u, \end{aligned}$$

where  $a, c > 0$  are constants independent of  $x \in M$  and  $v \in E_x^s$ ,  $w \in E_x^u$ . The subbundles  $E^s$  and  $E^u$  are called the *stable* and *unstable* subbundles. D. V. Anosov proved that they are integrable and that also  $E^s \oplus \mathbb{R}\xi$  and  $E^u \oplus \mathbb{R}\xi$  are integrable ([3]). Now let  $(M, \phi, \xi, \eta, G)$  be a contact metric manifold for which the Reeb vector field is Anosov. Then  $(E^s, E^u)$  defines a bi-Legendrian structure on  $M$ . Moreover, the invariance of  $E^s$  and  $E^u$  with respect to the flow can be expressed in terms of Legendrian foliations just by the flatness of  $(E^s, E^u)$ . Note that the paracontact metric structure induced on  $M$  is in fact para-Sasakian. Hence  $\xi$  is Killing with respect to the semi-Riemannian metric of the canonical paracontact metric structure induced on  $M$  by that bi-Legendrian structure, even if it can never be Killing with respect to the associated metric  $G$  in view of (4.4). The most notable example of a contact manifold for which the Reeb vector field is Anosov is the tangent sphere bundle of a negatively curved manifold.

**4.5. Contact metric  $(\kappa, \mu)$ -spaces.** Let  $(M, \phi, \xi, \eta, G)$  be a contact metric  $(\kappa, \mu)$ -space, that is a contact metric manifold satisfying

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)\bar{h}X - \eta(X)\bar{h}Y)$$

for some constants  $\kappa, \mu \in \mathbb{R}$ , where  $2\bar{h}$  denotes the Lie derivative of  $\phi$  in the direction of  $\xi$ . These manifolds have been introduced and deeply studied by D. E. Blair, T. Koufogiorgos and B. J. Papantoniou in [4]. The authors proved that necessarily  $\kappa \leq 1$  and  $\kappa = 1$  if and only if  $M$  is Sasakian. Then for  $\kappa < 1$  the structure is not Sasakian and  $M$  admits three mutually orthogonal integrable distributions  $\mathcal{D}(0) = \mathbb{R}\xi$ ,  $\mathcal{D}(\lambda)$  and  $\mathcal{D}(-\lambda)$  corresponding to the eigenspaces of  $\bar{h}$ , where  $\lambda = \sqrt{1 - \kappa}$ . Therefore  $(M, \eta, \mathcal{D}(\lambda), \mathcal{D}(-\lambda))$  is a bi-Legendrian manifold. The induced paracontact metric structure is integrable and thus the canonical paracontact and the bi-Legendrian connections coincide. Moreover by a result in [10] those connections parallelize also  $\phi$ ,  $\bar{h}$  and  $G$ . The standard example of a contact metric  $(\kappa, \mu)$ -space is given by the tangent sphere bundle of a Riemannian manifold of constant sectional curvature.

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